

Duality and Combinatorics of Long Strings in ADS_3

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The counting of long strings in ADS_3 , in the context of Type IIB string theory on $ADS_3 \times S^3 \times T^4$, is used to exhibit the action of the duality group $O(5,5;Z)$, and in particular its Weyl Subgroup $S_5 \bowtie Z_2$, in the non-perturbative phenomena associated with continuous spectra of states in these backgrounds. The counting functions are related to states in Fock spaces. The symmetry groups also appear in the structure of compactifications of instanton moduli spaces on T^4 .

1. Introduction

We study counting problems related to the phenomena of long strings in ADS3, in the context of type IIB string theory on $ADS_3 \times S^3 \times T^4$, which is one of the examples which enters the Maldacena conjecture [1]. This example is of special interest because the CFT dual is a tractable 2D CFT based on an orbifold $S^N(T^4)$. Various aspects of the operator algebra and correlation functions have been studied for example in [2][3][4][5][6]. An important issue that has to be understood better in order to make the orbifold CFT more useful is the precise map between the moduli space of the CFT and that of the spacetime theory. Significant steps in this direction have been made in [7][8][9]. Closely related to the issue of moduli is the issue of dualities, since the the duality group $O(5, 5; Z)$ appears in the description of the moduli space $O(5, 5; Z) \backslash O(5, 5; R) / O(5) \times O(5)$ of $R^6 \times T^4$ compactifications of type IIB. The ADS3 background is obtained by choosing a string in six dimensions and going to the near-horizon limit. The string can be a bound state of one or more of the following : D-string, D5-brane wrapped on the T^4 , D3-branes wrapped along the two cycles of the torus, NS one-brane (elementary string) or NS 5-brane wrapped on T^4 . The allowed charges of the string live are vectors in a lattice $\Gamma^{5,5}$ (described more explicitly in section 2). After choosing a vector in $\Gamma^{(5,5)}$ describing the background string it is often useful to focus on subgroup $O(4, 5; Z)$ of $O(5, 5 : Z)$, see for example [10][11]. For the kinds of questions we will be asking it will be interesting to look at subgroups of $O(5, 5 : Z)$ which may or may not belong to $O(4, 5; Z)$.

A very interesting class of phenomena in ADS3, not directly accessible from the orbifold model at its free point, was studied in [12][8]. This involves, in the simplest case, an ADS3 background obtained from the near horizon limit of Q_1 D1 branes and Q_5 D5-branes. This system allows a D1 (or D5) brane to split off and grow to infinity at finite cost in energy. A semiclassical calculation (valid for large Q_1, Q_5) of the split string worldvolume action shows that the string worldvolume theory is a Liouville theory, which is known to have a continuous spectrum above a threshold. The threshold is at $Q_5/4$ if we have a split D1 string, and $Q_1/4$ if we have a split D5-string, The spectrum of the Hamiltonian in ADS3 therefore has a continuum starting at $Q/4$ where $Q = \min(Q_1, Q_5)$. This formula has the very interesting symmetry under exchange of Q_1 and Q_5 , which is in $O(5, 5; Z)$ but not in $O(4, 5; Z)$.

These long string phenomena correspond to the splitting $(Q_1, Q_5) = (Q_1 - 1, Q_5) + (1, 0)$ or $(Q_1, Q_5) = (Q_1, Q_5 - 1) + (0, 1)$. The conditions for such splittings $Q = \sum q^{(i)}$

to be BPS were described in [8] in terms of the geometry of lattices and 5-planes and $R^{(5,5)}$ (we review this in section 2). For generic moduli there are no BPS splittings. An interesting class of splittings happens when the NS sector B-field moduli and the RR sector C-field moduli are set to zero (For the bulk of this paper we will work with the $B = C = 0$ condition, with off-diagonal components of the metric set to zero, and we get back to non-zero B, C in section 7). The same symmetry under exchange of Q_1 and Q_5 can be seen in the simple exercise of counting of the number of distinct splittings of the (Q_1, Q_5) system. The counting is not symmetric under exchange of (Q_1, Q_5) with $(Q_1 Q_5, 1)$.

In this paper we study generalizations of the counting of BPS splittings, and we find that the Weyl group of $O(5, 5; Z)$, which is generated by the symmetric group S_5 and a Z_2 , indicated by writing the Weyl group as $S_5 \rtimes Z_2$, has an interesting action on the counting functions associated with BPS splittings. It is natural to expect that it also is a symmetry of the more detailed dynamics of continuous spectra associated with such splittings. Section 2 is a review of relevant background. Sections 3-5 deal with splittings of different kinds of charges. Section 6 discusses the connections of these BPS brane separation problems and the associated symmetries with compactifications of instanton moduli spaces. Section 7 discusses some aspects of splittings beyond $B = C = 0$. Section 8 discusses the description from the gauge theory point of view of splittings involving NS charges from D-brane systems.

2. Review

2.1. Backgrounds as 5-planes

Choosing a background for IIB theory on $R^6 \times T^4$ requires choosing a 5-plane in $R^{(5,5)}$, modulo discrete identifications [13] (for related discussions see [11][10][14][15]). The 5-plane is spanned by vectors $E^{(0)}, E^{(1)}, E^{(2)}, E^{(3)}, E^{(4)}$. We use the notation of [11] (with a minor reshuffling of entries) for the vectors spanning the positive 5-plane Θ in $R^{(5,5)}$.

$$\begin{aligned}
E^\mu &= (v^\mu; -C.v^\mu, 0) \in R^{4,4} \times R^{(1,1)} \\
E^4 &= (0; \beta, 1) \in R^{4,4} \times R^{(1,1)} \\
v^i &= (-B.\omega^i, 0; \omega^i) \in R^{1,1} \times R^{(3,3)} \\
v^0 &= (\alpha, 1; 0) \in R^{1,1} \times R^{(3,3)}
\end{aligned} \tag{2.1}$$

Here μ runs from 0 to 4 and i runs from 0 to 3. We also have $\beta = \frac{1}{g_6^2} - \frac{1}{2}C.C$, and $\alpha = V - \frac{1}{2}B.B$.

The charges of strings live in a lattice $\Gamma^{(5,5)} \in R^{(5,5)}$. The charge Q has components $(Q_1, Q_5; Q_{12}, Q_{34}, Q_{13}, Q_{42}, Q_{14}, Q_{23}; N_1, N_5)$. Here (Q_1, Q_5) are the charges from D1 strings and wrapped D5 strings, and (N_1, N_5) are the charges from NS strings and wrapped NS 5-branes. The Q_{ij} are the charges of strings obtained by wrapping D3 branes on the (ij) cycle. The bilinear form on $\Gamma^{(5,5)}$ evaluated on two vectors $Q^{(1)} = (Q_1^{(1)}, Q_5^{(1)}; Q_{12}^{(1)}, Q_{34}^{(1)}, Q_{13}^{(1)}, Q_{42}^{(1)}, Q_{14}^{(1)}, Q_{23}^{(1)}; N_1^{(1)}, N_5^{(1)})$ and $Q^{(2)} = (Q_1^{(2)}, Q_5^{(2)}; Q_{12}^{(2)}, Q_{34}^{(2)}, Q_{13}^{(2)}, Q_{42}^{(2)}, Q_{14}^{(2)}, Q_{23}^{(2)}; N_1^{(2)}, N_5^{(2)})$, is

$$\begin{aligned} (Q^{(1)}, Q^{(2)}) &= Q_1^{(1)} Q_5^{(2)} + Q_1^{(2)} Q_5^{(1)} + Q_{12}^{(2)} Q_{34}^{(1)} + Q_{12}^{(1)} Q_{34}^{(2)} \\ &+ Q_{13}^{(1)} Q_{42}^{(2)} + Q_{13}^{(2)} Q_{42}^{(1)} + Q_{14}^{(1)} Q_{23}^{(2)} + Q_{14}^{(2)} Q_{23}^{(1)} + N_1^{(1)} N_5^{(2)} + N_1^{(2)} N_5^{(1)} \end{aligned} \quad (2.2)$$

For a rectangular torus (i.e off-diagonal components of the metric set to zero) and with vanishing B and C fields, we have

$$\begin{aligned} E^0 &= (V, 1; 0; 0, 0) \\ E^i &= (0, 0; \omega^i; 0, 0) \\ E^4 &= (0, 0; 0; \frac{1}{g_6^2}, 1) \end{aligned} \quad (2.3)$$

We can choose :

$$\begin{aligned} \omega^1 &= R_1 R_2 dx_1 \wedge dx_2 + R_3 R_4 dx_3 \wedge dx_4 \\ \omega^2 &= R_1 R_3 dx_1 \wedge dx_3 - R_2 R_4 dx_2 \wedge dx_4 \\ \omega^3 &= R_1 R_4 dx_1 \wedge dx_4 + R_2 R_3 dx_2 \wedge dx_3 \end{aligned} \quad (2.4)$$

R_i are the circumferences of the circles. x_i are variables with periodicity 1. With these formulae, and $T(v) = |v_+|$, we can recover the mass formulae of [7].

Since physical quantities are given by projections of vectors in $\Gamma^{(5,5)}$ to the 5-plane, the space of physically inequivalent vacua is obtained by modding out by the symmetries of $\Gamma^{(5,5)} : O(5, 5; Z) \backslash O(5, 5; R) / O(5, R) \times O(5, R)$.

When we choose a charge for the string in 6 dimensions living in $\Gamma^{(5,5)}$, having the property $Q^2 > 0$, the attractor equations [16] imply that the near horizon moduli satisfy the condition that Q is parallel to the 5-plane [10][8].

2.2. BPS splitting into multiple parts

The condition for a splitting $Q = q^{(1)} + q^{(2)}$ to be BPS can be expressed by saying that the projection of $q^{(1)}$ or $q^{(2)}$ to Θ is proportional to Q [8]. The projection has to be non-negative i.e $q \cdot Q \geq 0$.

The density of states in the continuum associated with the splittings into multiple summands will be larger than in the case of two summands. The condition for these splittings to be BPS can be obtained as in [8]. Consider a vector $Q \in \Gamma^{(5,5)}$ of charges which can be written as $Q = q^{(1)} + q^{(2)} + \dots + q^{(l)}$, satisfying the condition that the projections to the positive 5-plane obey $|Q_+| = |q_+^{(1)}| + |q_+^{(2)}| + \dots + |q_+^{(l)}|$. The near horizon geometry satisfies $Q_+ = Q$. These conditions can be satisfied if each vector $q^{(i)}$ in the decomposition has a projection to the 5-plane, $q_+^{(i)}$ which is parallel to Q .

If we start with a system of charges (Q_1, Q_5) and study its splittings at $B = C = 0$ with generic radii, we get a set of splittings sitting in a $\Gamma^{1,1}$ lattice. The symmetry of this lattice includes exchanging $(Q_1, Q_5) \rightarrow (Q_5, Q_1)$, but does not include $(Q_1, Q_5) \rightarrow (Q_1 Q_5, 1)$. The duality group $O(\Gamma^{5,5})$ does allow us to map the system (Q_1, Q_5) to both (Q_5, Q_1) and to $(Q_1 Q_5, 1)$. The first kind of map allows us to start from $B = C = 0$ and end with $B = C = 0$. The second does not [7]. There is also S-duality symmetry which is often discussed and exploited. It allows us to relate the physics of backgrounds containing NS-NS fields, to the physics of backgrounds with RR charge. It preserves the $B = C = 0$ condition.

We will study splittings with the condition $B = C = 0$, torus rectangular, and identify counting functions describing the splittings. These functions exhibit symmetries in the Weyl group, not too surprisingly since this is known to preserve these conditions on the moduli [17]. In section three we look at splittings of the charge vector $(Q_1, Q_5; \vec{0}; 0, 0)$ system. Section 4 deals with the splittings of $(Q_1, Q_5; Q_{12}, Q_{34}, 0, 0, 0, 0; 0, 0)$ system. Section 5 describes splittings of $(Q_1, Q_5; Q_{12}, Q_{34}, Q_{13}, Q_{42}, Q_{14}, Q_{23}; N_1, N_5)$.

2.3. Liouville theory on long string

A Liouville description of long strings was given in [8]. Further discussion from a 2D gauge theory point of view appeared in [18]. We have a pair of branes, one with charge q and the other with charge $Q - q$ splitting from one with charge Q . The tensions of the split strings also add up to the tension of the string of charge Q . A Liouville theory on the long string was derived

$$S = T r_0^2 \int \sqrt{g} ((\partial \Phi)^2 + \Phi R) \quad (2.5)$$

where T is the tension of the brane and r_0 is the radius of AdS_3 . The Liouville field Φ is related to the radial direction. After rescaling the field, the above action is a Liouville

action having the background charge $\mathcal{Q} = \sqrt{4\pi T r_0^2}$. The conformal field theory has a central charge $c_{Liouville} = 3\mathcal{Q}^2$ and a massgap $\Delta_0 = \frac{\mathcal{Q}^2}{8}$. We assume now that $Q^2 \gg q^2$ and that the AdS geometry is given by Q . Using the same arguments leading to (2.5) we obtain using $T(q) = qQ/|Q|$ and $r_0^2 = 2\pi|Q|$ that $\mathcal{Q} = 2qQ$. The previous expression reduces to the expected one in the case of q being a $D1$ string, i.e giving a central charge of $6Q_5 = 6(Q_1 Q_5) - 6(Q_1 - 1)(Q_5)$.

In general, we may expect the string of charge q to see a geometry given by $Q - q$. Interpreting the Liouville theory as living on an “interaction string” whose central charge is the difference between $3Q^2$ and $3(Q - q)^2 + 3q^2$, we have

$$\mathcal{Q} = 2q(Q - q), \quad c_{Liouville} = 6q(Q - q) \quad (2.6)$$

Using the finite Q_5 result in [8] we would expect by U-duality that a split string of charge q would lead to central charges, and thresholds given by :

$$\begin{aligned} \mathcal{Q} &= 2q(Q - q), \\ c &= 3\mathcal{Q}^2, \\ \Delta_0 &= \frac{(\mathcal{Q} - 1)^2}{4Q}. \end{aligned} \quad (2.7)$$

3. Counting of splittings of $(Q_1, Q_5; \vec{0}; 0, 0)$

We describe the BPS splittings of the system $(Q_1, Q_5; 0; 0, 0)$. We are always working with $B = C = 0$ and rectangular tori, unless explicitly specified otherwise. We first consider generic radii, and then rational radii, where we further specialize to $R_i = g_6^2 = 1$.

3.1. Splittings of $(Q_1, Q_5; \vec{0}; 0, 0)$ for generic radii.

We would like to perform a detailed counting of the splittings for a given charge. Consider the charge $(N, 1)$. We can split it as follows :

$$(N, 1) = (q, 1) + (q', 0) \quad (3.1)$$

where $q + q' = N$. The possible choices of q range from 0 to $N - 1$, so there are N of them. For small q' such BPS splittings lead to a continuum of states as explained in [8]. There are also splittings where we decompose $(N, 1)$ into a sum of more than two vectors. These will also contribute to the Hamiltonian for string theory on $ADS3$ a continuum of states

(with the higher density of states associated with a multistring system as opposed to a two-string system).

To formulate physically sensible counting rules we have to decide whether a subsystem like $(1, 0) + (1, 0)$ should be counted as identical to $(2, 0)$ or as different. The $(1, 0)$ system with unit D1-charge can be mapped to an elementary string. Since we understand the perturbative multi-string Hilbert space, we know that in the sector with winding number two, we can have two singly wound states or a doubly wound state. So $(1, 0) + (1, 0)$ should be counted differently. Similarly we can argue that a string with charges $(2, 2)$ should be counted as distinct from a string $(1, 1) + (1, 1)$. By duality this is related to the fact that with winding number 2 and momentum 2, we can have a single string with these quantum numbers or two separate strings with these quantum numbers.

We have two kinds of splittings of the charge vector $(N, 1)$. The first has the form $(N, 0) = (0, 1) + (N_1, 0) + \cdots (N_l, 0)$, where N is being partitioned into l non-zero parts $N = N_1 + N_2 + \cdots N_l$. The second kind of splitting takes the form $(N, 0) = (N_1, 1) + (N_2, 0) + \cdots (N_l, 0)$. For each partition of N we can place the 1 in a number of different ways, but if the partition contains some integer more than once, placing the 1 next to different copies of the integer should not be counted more than once. As we see in the next paragraph this is automatically taken into account by a Fock space description in terms of bosonic oscillators. Equivalently for each partition of N , there are k different ways of placing the 1 where k is the number of distinct integers in the partition. So the total number of splittings of the second kind is equal to the sum of k over all the partitions of N .

This counting can be written in terms of a function

$$\tilde{P}(x, t) = \prod_{l=1}^{\infty} \frac{1}{(1-x^l)} \frac{1}{(1-tx^l)} \quad (3.2)$$

Let $P(x, t) = \sum_{N,m=0}^{\infty} P(N, m) x^N t^m$. The number of splittings is

$$N_s(N, 1) = P(N, 0) + P(N, 1) \quad (3.3)$$

Note that $P(N, 0)$ is just the number of partitions of N , and it counts the number of splittings of the form $(N, 1) = (0, 1) + (N_1, 0) + (N_2, 0) + \cdots (N_k, 0)$. The second term $P(N, 1)$ counts the number of splittings of the form $(N, 1) = (N_1, 1) + (N_2, 0) + \cdots (N_k, 0)$. This counting can be described in Fock space language. Consider the oscillators $\alpha_{-k,l}$

where k can take values from 1 to N , and l is a discrete index which can take values 0 or 1. $P(N, 0)$ counts the number of states in a Fock space where all the l take the value 0, and the oscillator indices k add up to N .

$$\alpha_{-N_1,0} \cdots \alpha_{-N_k,0} |0\rangle \quad (3.4)$$

$P(N, 1)$ counts the number of states where the l values add up to 1, which is the same as saying that one of the l values takes the value 1. One of these states is :

$$\alpha_{-N_1,1} \cdots \alpha_{-N_k,0} |0\rangle \quad (3.5)$$

The bosonic statistics of the α oscillators guarantees that associating the 1 with different copies of the same N_i overcounted. A more economical way of writing the number of splittings is to use the generating function :

$$P_1(x_1, x_2) = \prod_{k,l} \frac{1}{(1 - x_1^k x_2^l)} \quad (3.6)$$

where k and l extend from 0 to ∞ , except that $k = l = 0$ is not allowed. Define the coefficient $P_1(n, m)$ as the coefficients appearing in the expansion :

$$P_1(x_1, x_2) = \sum_{n,m} P_1(n, m) x_1^n x_2^m \quad (3.7)$$

The number of splittings is just $P_1(N, 1)$. It is clear that $P_1(N, 1)$ is equal to $P_1(1, N)$.

We can also count the number of ways of splitting a more general charge (Q_1, Q_5) . We can write the number of distinct splittings $N_s(Q_1, Q_5)$ in terms of the coefficients of this quantity as follows:

$$N_s(Q_1, Q_5) = P_1(Q_1, Q_5) \quad (3.8)$$

This number is equal to the number of states in a Fock space of the form

$$\alpha_{-(k_1,l_1)} \alpha_{-(k_2,l_2)} \cdots \alpha_{-(k_m,l_m)} |0\rangle \quad (3.9)$$

where the integers k_1 through k_m add up to Q_1 and the integers l_1 through l_m add up to Q_5 . Oscillators of the form $\alpha_{-(k,0)}$ or $\alpha_{-(0,l)}$ are allowed but $\alpha_{-(0,0)}$ is not allowed. It is clear that we have the symmetry :

$$N_s(Q_1, Q_5) = N_s(Q_5, Q_1). \quad (3.10)$$

To make the counting even more explicit, we can restrict to splittings into two parts only, to get for the system $(Q_1 Q_5, 1)$ a number $Q_1 Q_5$. For the system (Q_1, Q_5) we get $\frac{(Q_1+1)(Q_5+1)}{2} - 1$. These formulae clearly show that there is a symmetry under exchange of Q_1 and Q_5 , but no symmetry under exchange of (Q_1, Q_5) into $(Q_1 Q_5, 1)$.

Note that in all of the above counting we have not included splittings of the form $(Q_1, Q_5) = (-q'_1, -q'_5) + (q_1 + q'_1, q_5 + q'_1)$, with q'_1, q'_5 positive. From the connection to instanton moduli spaces discussed in a later section, this is a natural restriction to consider. For a splitting of this form, when q'_1, q'_5 are positive and small, the arguments of [8] show that the energy required to create such a long string is infinite and the system is not BPS. A duality invariant way of stating this restriction is that we are considering splittings $Q = q^{(1)} + \dots + q^{(l)}$ where the projections of $q^{(i)}$ to the 5-plane are not only proportional to Q but that the constant of proportionality is positive i.e $q^{(i)} \cdot Q \geq 0$.

3.2. Refinements of counting

Note that to keep the formulae simple, we have counted splittings by merely counting the number of distinct string charges that can appear in the splittings of a given charge. One may also weight the splittings by the number of distinct states associated with the ground states of the split strings. If we choose to do such a counting, we should have, for a charge (k, l) , a multiplicity of $p_{16}(kl)$, where $p_{16}(kl)$ is the number of states at level kl in a Fock space with 16 distinct oscillators. Such a counting can be done by modifying the above partition functions as follows : We replace in (3.6) the expression $\frac{1}{(1-x_1^k x_2^l)}$ by $\frac{1}{(1-x_1^k x_2^l)^{p_{16}(kl)}}$. A further refinement would involve defining a partition function which sums the lowest energy states coming from each splitting by weighting with an energy dependent exponential. This requires further dynamical information on the energy associated with each splitting. The simplest counting problem we have considered, which we extend to systems with more charges in the subsequent sections, suffices for the purpose at hand which is to exhibit the appropriate subgroups of the duality group.

3.3. Splittings of $(Q_1, Q_5; \vec{0}; 0, 0)$ at special radii.

Consider the splitting of a

$$\begin{aligned} (Q_1, Q_5; 0, 0, 0, 0, 0, 0; 0, 0) &\rightarrow q + (Q - q); \\ q &= (q_1, q_5, q_{12}, q_{34}, 0, 0, 0, 0; 0, 0) \end{aligned} \tag{3.11}$$

According to [8], this is a BPS splitting if the projection of q to the 5-plane is proportional to (Q_1, Q_5) . Vanishing of the projection along E^1 leads to the condition

$$\frac{-q_{12}}{q_{34}} = \frac{R_1 R_2}{R_3 R_4} \quad (3.12)$$

This means that such splitting can only happen when $\frac{R_1 R_2}{R_3 R_4}$ is a rational number. While the small instanton singularity discussed in [8] requires tuning B -fields, these singularities require tuning geometrical moduli. The fact that we need to tune the B -field to zero could be understood from properties of instantons because the instantons are no longer point-like for non-zero generic B -fields.

It is interesting that these conditions on the geometry can be partially understood from properties of instanton moduli spaces. The splitting should correspond to instanton solutions where the connection takes block diagonal form :

$$A = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ & A^{(1)} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & A^{(2)} & \cdot \end{pmatrix} \quad (3.13)$$

Here $A^{(1)}$ is a connection in $U(q_5)$ with instanton number q_1 , and fluxes

$$\int \frac{1}{2\pi} \text{tr} F_{12} = q_{34}, \quad \int \frac{1}{2\pi} \text{tr} F_{34} = q_{12}$$

$A^{(2)}$ is a $U(Q_5 - q_5)$ connection with instanton number $Q_1 - q_1$, and fluxes $-q_{12}, -q_{34}$. While such block diagonal configurations can exist for any radii, it is interesting that when the rationality conditions (3.12) are satisfied, some special instantons are possible. For special cases where $q_1 q_5 - q_{12} q_{34} = 0$, we can realize this as a $U(q_5) \subset U(Q_5)$ instanton configuration built from constant field strengths F_{12}, F_{34} . The flux quantization conditions combined anti-self duality, lead to the rationality conditions on the radii :

$$\begin{aligned} F_{12} R_1 R_2 &= q_{34} \\ F_{34} R_3 R_4 &= q_{12} \\ F_{12} &= -F_{34} \\ \Rightarrow \frac{R_1 R_2}{R_3 R_4} &= -\frac{q_{34}}{q_{12}} \end{aligned} \quad (3.14)$$

Such constant field strength solutions (torons) were written down in [19] and their connection to D-brane bound states studied in [20][21]. Torons are in this sense closely related to splittings involving $q^2 = 0$.

More generally we can have

$$(Q_1, Q_5; 0, 0, 0, 0, 0, 0; 0, 0) \rightarrow (q_1, q_5; q_{12}, q_{34}, q_{13}, q_{42}, q_{14}, q_{23}; n_1, n_5) + \dots \quad (3.15)$$

If all the radii are adjusted to the self-dual value, and the six-dimensional coupling also adjusted to 1 : we now have four independent parameters which can be arbitrary integers, $(q_{12}, q_{13}, q_{14}, n_1)$. The remaining parameters are determined by

$$\begin{aligned} q_{34} &= -q_{12} \\ q_{42} &= -q_{13} \\ q_{23} &= -q_{14} \\ n_5 &= -n_1 \end{aligned} \quad (3.16)$$

The norm of the first vector is equal to $2(q_1 q_5 - q_{12}^2 - q_{13}^2 - q_{14}^2 - n_1^2)$. The BPS condition requires that this is greater or equal to zero. When we realize these configurations in gauge theory, we have a condition $q_5 > 0$. The condition $q^2 \geq 0$ means that $q_1 \geq 0$. Since the q_1 sum to Q_1 and the q_5 sum to Q_5 , we have $0 \leq q_1 \leq Q_1$ and $0 \leq q_5 \leq Q_5$. The condition $q^2 \geq 0$ then only allows a finite number of solutions.

The splittings of this form can be counted using generating functions :

$$P_2(x_1, x_2, x_3, x_4, x_5, x_6) = \prod_{k,l,m,n,p,r} \frac{1}{(1 - x_1^k x_2^l x_3^m x_4^n x_5^p x_6^r)} \quad (3.17)$$

The product is constrained by

$$\begin{aligned} k &\geq 0, l \geq 0 \\ kl - m^2 - n^2 - p^2 - r^2 &\geq 0 \end{aligned} \quad (3.18)$$

Defining $P(n_1, n_2, n_3, n_4, n_5, n_6)$ as the coefficient of $x_1^{n_1} x_2^{n_2} x_3^{n_3} x_4^{n_4} x_5^{n_5} x_6^{n_6}$ in $P(x_1, x_2, x_3, x_4, x_5, x_6)$, the desired counting function is $P(Q_1, Q_5, 0, 0, 0, 0)$.

At the special radii and g_6 we have a large class of splittings involving strings with extra 3-anti3 charges and NS1-5 charges $(q_1, q_5; q_{12}, q_{34}, q_{13}, q_{42}, q_{14}, q_{23}; n_1, n_5)$. We saw that the charges $q_{34}, q_{42}, q_{23}, n_5$ are determined by the charges $q_{12}, q_{13}, q_{14}, n_1$ respectively. There is an $S_4 \rtimes Z_2$ subgroup of the Weyl group which acts on the charges of these split

strings. The S_4 just permutes the four independent charges and the Z_2 acts by a reflection $q_{12} \rightarrow -q_{12}$. The counting function $P(Q_1, Q_5)$ of course has the symmetry of exchanging Q_1 and Q_5 . A $Z_2 \times (S_4 \bowtie Z_2)$ subgroup of $(S_5 \bowtie Z_2)$ is therefore manifest here. The first factor is a symmetry acting on the charge of the string we start with i.e exchanging Q_1, Q_5 . The second is a symmetry acting on the strings that can appear in the splitting. By choosing $R_i = g_6^2 = 1$ we made sure that the $S_4 \bowtie Z_2$ acts as a symmetry of the Hamiltonian for the corresponding ADS background. For more general radii satisfying the rationality conditions, a split string for one geometrical modulus is mapped to a split string at another geometrical modulus.

4. Splittings of a system $(Q_1, Q_5; Q_{12}, Q_{34}, \vec{0}; 0, 0)$

Consider first the splittings of

$$(Q_1, Q_5, Q_{12}, Q_{34}) = (q_1, q_5; q_{12}, q_{34}, 0, 0, 0, 0; 0, 0) + \dots \quad (4.1)$$

We again consider rectangular tori with $B = C = 0$. The condition on the near horizon geometry takes the form :

$$\begin{aligned} \frac{Q_1}{Q_5} &= V \\ \frac{Q_{12}R_1R_2}{Q_{34}R_3R_4} &= 1 \end{aligned} \quad (4.2)$$

The condition that the projection of the vector q to the five-plane is parallel to Q takes the form :

$$\frac{(q_1Q_5 + q_5Q_1)}{Q_1Q_5} = \frac{(q_{12}Q_{34} + q_{34}Q_{12})}{(Q_{12}Q_{34})} \quad (4.3)$$

If we consider splittings of the more general form :

$$(Q_1, Q_5, Q_{12}, Q_{34}) = (q_1, q_5, q_{12}, q_{34}, q_{13}, q_{42}, q_{14}, q_{23}, n_1, n_5) + \dots \quad (4.4)$$

we still have the above conditions and we have restrictions on the moduli of the form :

$$\begin{aligned} \frac{q_{13}R_1R_3}{q_{42}R_4R_2} &= -1 \\ \frac{q_{14}R_1R_4}{q_{23}R_3R_2} &= -1 \\ \frac{n_1g_6^2}{n_5} &= -1 \end{aligned} \quad (4.5)$$

Let us first consider splittings of the first kind. These can be counted as the coefficient of $x_1^{Q_1} x_2^{Q_5} x_3^{Q_{12}} x_4^{Q_{34}}$ in the series

$$P(x_1, x_2, x_3, x_4) = \prod_{q_1, q_5, q_{12}, q_{34}} \frac{1}{(1 - x_1^{q_1} x_2^{q_5} x_3^{q_{12}} x_4^{q_{34}})} \quad (4.6)$$

Here the q 's obey the conditions in (4.3), in addition to the usual $q^2 \geq 0$ and $q \cdot Q \geq 0$. A class of solutions of these conditions (we haven't proved that they give a complete set of solutions) can be written as

$$q = \lambda_1 q_{(1)} + \lambda_2 q_{(2)} \quad (4.7)$$

where $q_{(1)} = (\frac{Q_1}{\lambda_L}, 0; \frac{Q_{12}}{\lambda_L}, 0, 0, 0, 0, 0; 0, 0)$ and $q_{(2)} = (0, \frac{Q_5}{\lambda_R}, 0; 0, \frac{Q_{34}}{\lambda_R}, 0, 0, 0, 0; 0, 0)$. Here

$$\lambda_L = \gcd(Q_1, Q_{12}); \lambda_R = \gcd(Q_5, Q_{34})$$

and

$$0 \leq \lambda_1 \leq \lambda_L; 0 \leq \lambda_2 \leq \lambda_R.$$

Splittings coming from this class of solutions can be counted as the coefficient of $x_1^{\lambda_L} x_2^{\lambda_R}$ in the series :

$$\prod_{\lambda_1, \lambda_2} \frac{1}{(1 - x_1^{\lambda_1} x_2^{\lambda_2})} \quad (4.8)$$

This counting function $N_s(Q_1, Q_5, Q_{12}, Q_{34})$ is symmetric under a group $S_2 \bowtie Z_2$. The S_2 exchanges the (Q_1, Q_5) pair with the (Q_{12}, Q_{34}) pair. The Z_2 generator can be taken to be the exchange of (Q_1, Q_5) to (Q_5, Q_1) .

As in the previous section we can proceed to the case where the other ratios of radii and the coupling are adjusted according to (4.5) and we can further specialize to the case where all $R_i = 1 = g_6^2$, to find a group $S_3 \bowtie Z_2$ acting on the allowed extra charges.

5. Splittings of a system $(Q_1, Q_5, Q_{12}, Q_{34}, Q_{13}, Q_{42}, Q_{14}, Q_{23}, N_1, N_5)$

The condition that this charge be parallel to the 5-plane gives :

$$\begin{aligned} \frac{Q_1}{Q_5 V} &= 1 \\ \frac{Q_{12} R_1 R_2}{Q_{34} R_3 R_4} &= 1 \\ \frac{Q_{13} R_1 R_3}{Q_{42} R_2 R_4} &= 1 \\ \frac{Q_{14} R_1 R_4}{Q_{23} R_2 R_3} &= 1 \\ \frac{N_1 g_6^2}{N_5} &= 1 \end{aligned} \quad (5.1)$$

Consider splittings of the form $Q \rightarrow q + \dots$, where the charge q has components $(q_1, q_5; q_{12}, q_{34}, q_{13}, q_{42}, q_{14}, q_{23}; n_1, n_5)$. We have conditions $q^2 \geq 0$, and requiring that the projection of q to the 5-plane be parallel to Q leads to conditions :

$$\begin{aligned} \frac{(q_1 Q_5 + q_5 Q_1)}{Q_1 Q_5} &= \frac{(q_{12} Q_{34} + q_{34} Q_{12})}{Q_{12} Q_{34}} = \frac{(q_{13} Q_{42} + q_{42} Q_{13})}{Q_{13} Q_{42}} \\ &= \frac{(q_{14} Q_{23} + q_{23} Q_{14})}{Q_{14} Q_{23}} = \frac{(n_1 N_5 + n_5 N_1)}{N_1 N_5} \end{aligned} \quad (5.2)$$

A class of solutions to these equations takes the form

$$\begin{aligned} \lambda_1 q^{(1)} + \lambda_2 q^{(2)}, \\ 0 \leq \lambda_1 \leq \lambda_L \\ 0 \leq \lambda_2 \leq \lambda_R \end{aligned} \quad (5.3)$$

where $\lambda_L = \gcd(Q_1, Q_{12}, Q_{13}, Q_{14}, N_1)$ and $\lambda_R = \gcd(Q_5, Q_{34}, Q_{42}, Q_{23}, N_5)$, and $q^{(1)} = \frac{1}{\lambda_L}(Q_1, 0, Q_{12}, 0, Q_{13}, 0, Q_{14}, 0, N_1, 0)$ and $q^{(2)} = \frac{1}{\lambda_R}(0, Q_5, 0, Q_{34}, 0, Q_{42}, 0, Q_{23}, 0, N_5)$. The number of these splittings $N_s(Q_1, Q_5, Q_{12}, Q_{34}, Q_{13}, Q_{42}, Q_{14}, Q_{23}; N_1, N_5)$ can be counted using the coefficient of $x_1^{\lambda_L} x_2^{\lambda_R}$ in the following series:

$$\prod_{\lambda_1, \lambda_2} \frac{1}{(1 - x_1^{\lambda_1} x_2^{\lambda_2})}, \quad (5.4)$$

where $\lambda_1 \geq 0, \lambda_2 \geq 0$, but $(\lambda_1, \lambda_2) = (0, 0)$ is excluded. The function $N_s(Q_1, Q_5, Q_{12}, Q_{34}, Q_{13}, Q_{42}, Q_{14}, Q_{23})$ is invariant under $S_5 \rtimes Z_2$. The Z_2 can be taken to be the exchange of (Q_1, Q_5) to (Q_5, Q_1) . The S_5 permutes the five blocks : $(Q_1, Q_5), (Q_{12}, Q_{34}), (Q_{13}, Q_{42}), (Q_{14}, Q_{23})$ and (N_1, N_5) .

6. Beyond $B = C = 0$.

The duality group $O(5, 5 : Z)$ can be used to map the charges $(Q_1, Q_5, 0, 0)$ to the charge $(Q_1 Q_5, 1, 0, 0)$. An $O(2, 2)$ subgroup of the $O(5, 5)$ will suffice to do that, as pointed out by [7]:

$$\begin{pmatrix} adQ_5^2 & -bcQ_1^2 & acQ_1Q_5 & bdQ_1Q_5 \\ -bc & ad & -ac & -bd \\ abQ_5 & -abQ_1 & a^2Q_5 & b^2Q_1 \\ cdQ_5 & -cdQ_1 & c^2Q_5 & d^2Q_1 \end{pmatrix} \quad (6.1)$$

Here a, b, c, d have been chosen to satisfy the condition $adQ_5 - bcQ_1 = 1$. In fact there exists a choice of $c = d = 1$ which leads to

$$\begin{pmatrix} aQ_5^2 & -bQ_1^2 & aQ_1Q_5 & bQ_1Q_5 \\ -b & a & -a & -b \\ abQ_5 & -abQ_1 & a^2Q_5 & b^2Q_1 \\ Q_5 & -Q_1 & Q_1 & Q_5 \end{pmatrix} \quad (6.2)$$

. These matrices allow us to see explicitly that these transformations which violate the $B = C = 0$ condition.

We can use this map start with the splittings of a (Q_1, Q_5) system to get splittings of a $(Q_1Q_5, 1)$ system. The splittings of $(Q_1, Q_5) = (4, 3)$ are studied here. This can be mapped by $O(5, 5; Z)$ to the charge $(12, 1)$. A first class of splittings of $(4, 3)$ involving no extra charges is :

$$\begin{aligned} (4, 3) &\rightarrow (4, 0) + (0, 3) \\ &\rightarrow (4, 1) + (0, 2) \\ &\rightarrow (4, 2) + (0, 1) \\ &\rightarrow (3, 0) + (1, 3) \\ &\rightarrow (3, 1) + (1, 2) \\ &\rightarrow (3, 2) + (1, 1) \\ &\rightarrow (3, 0) + (1, 0) \\ &\rightarrow (2, 0) + (2, 3) \\ &\rightarrow (2, 1) + (2, 2) \end{aligned} \quad (6.3)$$

We also have 12 splittings of the charge $(12, 1)$ into different charges of the form $(k, 1) + (12 - k, 0)$ where k ranges from 11 to 0. When these are mapped to splittings of the $(4, 3)$ system using $O(5, 5; Z)$ (in fact $O(2, 2; Z)$ suffices), we get a bunch of splittings of

$$(4, 3) \rightarrow (16 - k, k - 9, k - 12, k - 12) + (k - 12, 12 - k, 12 - k, 12 - k) \quad (6.4)$$

This shows that the class of counting functions etc. that we have described can be used to obtain information about splittings at moduli which go beyond the $B = C = 0$ condition. In general the B, C values have explicit dependence on Q_1 and Q_5 , and such dependence will show up in the coefficients of marginal operators necessary to go from the free point of the orbifold description in terms of $S^{Q_1Q_5}(T)$ to the point which has a simple description in terms of (Q_1, Q_5) system [7].

The data associated with splittings and thresholds etc. seems best viewed as living on the space

$$\Gamma^{5,5} \times O(5, 5; R)/O(5) \times O(5), \quad (6.5)$$

with a manifest $O(5, 5; Z)$ symmetry which acts on both charges and moduli. It will be interesting to see if the class of splittings at $B = C = 0$, a large class of which we have considered in this paper, exhausts all the physics of long strings that one encounters as one moves in the space (6.5).

7. Instantons

Properties of instanton moduli spaces have come up in the discussions of the splittings above. Some aspects are developed in more detail here, and some puzzling features discussed.

First consider the splittings of (Q_1, Q_5) which do not involve extra 3-brane charges. They correspond to a decomposition of the compactified moduli space of instantons. We will discuss some features¹ of the Uhlenbeck compactification [22].

The compactified moduli space of instantons on T^4 for gauge group $U(r)$ and instanton number k , $\bar{\mathcal{M}}_{(r,k)}$, admits a double stratification, labelled by two integers (f, p) which count the number of $U(1)$ flat connections and the number of point-like instantons respectively. T-duality inverts the volume, exchanges rank and instanton number, and flips the integers f and p .

$$\bar{\mathcal{M}}_{r,k} = \bigcup_{(f,p)} (\mathcal{M}_{r-f,k-p}^{(0)} \times S^p(T) \times S^f(T^*)) \quad \bigcup (S^k(T) \times S^r(T^*)) \quad (7.1)$$

The Moduli spaces $\mathcal{M}^{(0)}$ appearing in the above involve no flat connections and no point-like instantons. The integer f extends from 0 to $r-2$ and p extends from 0 to $k-2$. Naively one may have allowed them to extend to $r-1$ or $k-1$, but $\mathcal{M}_{1,l}$ and $\mathcal{M}_{l,1}^{(0)}$ are empty for any l . In fact such strata would be expected if we want a compactification which knows about all the possible split strings, including those of charge $(l, 1)$ as explained below.

Let us consider first the case where r, k are large, and f, p are small. The symmetric product $S^p(T)$ describes p long strings moving on T . The sigma model on this stratum contains sectors parametrized by partitions of p which describe different ways of partitioning

¹ We thank George Daskalopoulos for instructive discussions on this subject

the p long D1-strings into bound states. Similarly the symmetric product $S^f(T)$ contains sectors describing f long D5-strings. A stratum parametrized by (p, f) contains the physics of splittings of the type $(Q_1, Q_5) \rightarrow (Q_1 - f, Q_5 - p) + (f_1, 0) + \cdots + (f_l, 0) + (0, p_1) + \cdots + (0, p_k)$ where $p = p_1 + p_2 + \cdots + p_k$ and $f = f_1 + f_2 + \cdots + f_l$ are partitions of p and f . Geometrical structures corresponding to (f, p) bound states with both f and p non-zero do not seem to appear. Another puzzle appears when we allow p to be comparable to Q_5 . When p is equal to $Q_5 - 1$ we do not have the corresponding stratum, whereas the algebraic calculation of allowed BPS splittings continues to allow such splittings. Perhaps other compactifications, e.g the Gieseker compactification, do include the extra strata which would give a more detailed correspondence between strata and splittings.

As discussed in section 3.2, when we consider splittings involving extra 3-brane charges we need to consider reducible connections, i.e of block diagonal form where each block contains non-zero flux, but the fluxes add up to zero. When the original system contains extra 3-brane charges, we need to start with a bundle of rank Q_5 instanton number Q_1 and magnetic fluxes $\epsilon^{ijkl} Q_{kl}$. It is intriguing that in these cases a point-like instanton which should continue to exist in the classical instanton moduli space, does not correspond to any splitting, since the BPS splitting conditions require the split string to have more than just D1-charge. Perhaps some quantum corrected instanton moduli space exhibits the right stratifications.

8. Splitting NS charges from D-brane systems

The splittings of the (Q_1, Q_5) system which involve extra charges like three branes are easily described in the gauge theory of $U(Q_5)$ which contains magnetic fluxes allowing the description of the appropriate splittings. But the splittings which involve extra charges NS1-NS5 seem to be harder to describe. The NS1-charges are of course described by electric fluxes. We don't know how to describe the NS5-charges as a flux in the D-brane gauge theory. This is the famous transverse 5-brane problem in attempts to use Yang Mills as Matrix theory for compactifications on tori of dimension larger than 4. The transverse 5-brane problem is solved by appealing to little string theory [23] but available descriptions of this theory do not allow an explicit description of these splittings which generalize in a simple way the description in terms of point-like instantons, flat connections, block-diagonal connections that we have described. A description which stays within the confines of moduli spaces of six-dimensional gauge theories, can be given at the expense

of giving up the restriction to a fixed rank gauge group. The following remarks have some analogies to developments on duality in Matrix theory appearing in [24][25][26].

An example of a splitting which involves the NS1-NS5 charges is

$$(Q_1, Q_5; \vec{0}; 0, 0) \rightarrow (q_1, q_5; \vec{0}; 1, -1) + (Q_1 - q_1, Q_5 - q_5; \vec{0}; -1, 1) \quad (8.1)$$

For such a splitting to exist at $B = C = 0$, we need to adjust the six-dimensional coupling $g_6^2 = 1$. One way to give a gauge theoretic description of such a system is to map the charge $(q_1, q_5; \vec{0}; 1, -1)$ to $(\tilde{q}_1, \tilde{q}_5; \vec{0}; 0, 0)$, with $\tilde{q}_1 \tilde{q}_5 = q_1 q_5 - 1$, at the cost of turning on some non-trivial RR and/or B-fields. But this is a system which we can describe using $U(\tilde{q}_5)$ gauge theory with instanton number \tilde{q}_1 , with non-trivial couplings turned on due to the background fields. Similarly we can map $(Q_1 - q_1, Q_5 - q_5; \vec{0}; -1, 1)$ to (q'_1, q'_5) with $q'_1 q'_5 = (Q_1 - q_1)(Q_5 - q_5) - 1$, obtaining a $U(q'_5)$. Generically $q'_5 + \tilde{q}_5 \neq Q_5$, so we cannot think of this in terms of something happening within $U(Q_5)$ gauge theory. It is intriguing that whereas all the other splitting processes could be understood, at least roughly, in terms of some appropriate compactification of instanton moduli spaces, we here have to go beyond gauge theory at a fixed rank. It should be interesting to explore the systematic use of gauge theories of arbitrary rank to understand the full physics of the (Q_1, Q_5) system, using duality along the above lines.

9. Summary and Outlook

We found that counting functions related to the number of splittings of a string in six dimensions contains interesting information on symmetries associated with the phenomena of long strings in ADS3 and the associated continuums of the spectrum. The Weyl subgroup of $O(5, 5 : Z)$ played an interesting role.

The nature of the BPS splittings also gives information about the structure of the compactifications of instanton moduli spaces. The symmetries of BPS splittings are related to symmetries acting on strata of the compactifications. Some puzzles were raised regarding the details of this correspondence. It would be very interesting if these puzzles can be solved and the Fock space structures we described can be derived from appropriate compactifications of (possibly α' and g_s -corrected) instanton moduli spaces, thus generalizing the familiar relations between Fock spaces and instanton moduli spaces [27].

When we consider splittings of the (Q_1, Q_5) system which include vectors carrying non-trivial NS charges, we need to go beyond conventional compactifications of instanton

moduli spaces to even get a rough gauge theoretic understanding of the splittings. One avenue is to use facts about the duality group $O(5,5;Z)$ to show that we may need to go beyond instanton moduli spaces for bundles of a fixed rank to get a full gauge theory understanding of the (Q_1, Q_5) system.

In the course of exploring the relation between BPS splittings and instanton moduli spaces, we found that some such splittings, involving $q^{(i)}$ with $(q^{(i)})^2 = 0$ show a close relation to torons. It would be interesting to see if any special instanton solutions corresponding to splittings when $(q^{(i)})^2 > 0$. While we focused on these issues in the context of type IIB on T^4 , it would clearly be worthwhile to pursue similar questions in the case of $K3$, perhaps orbifold $K3$'s to begin with.

Whereas we used the counting of long strings to identify symmetries of the physics of long strings. It will be interesting to explore the consequences of these symmetries for the detailed dynamics, e.g the thresholds where the density of continuum states jumps. The counting problems themselves allow refinements mentioned in section 3. It would be interesting to explore if with these refinements, and with generalization to the three-charge system relevant to black hole entropy [28][29], one can gain new insights into the statistics of black holes.

Acknowledgements: We are happy to thank for discussions George Daskalopoulos, Zack Guralnik, Pei Ming Ho, Antal Jevicki, David Lowe, Juan Maldacena, Andy Strominger. This research was supported by DOE grant DE-FG02/19ER40688-(Task A).

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